PROJECTIVE CELL MODULES OF FROBENIUS CELLULAR ALGEBRAS

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Abstract. For a finite dimensional Frobenius cellular algebra, a sufficient and necessary condition for a simple cell module to be projective is given. A special case that dual bases of the cellular basis satisfying a certain condition is also considered. The result is similar to that in symmetric case.

1. Introduction

Cellular algebras were introduced by Graham and Lehrer [4] in 1996, motivated by previous work of Kazhdan and Lusztig [5]. They were defined by a so-called cellular basis satisfying certain axioms. The theory of cellular algebras provides a systematic framework for studying the representation theory of many interesting algebras from mathematics and physics. The classical examples of cellular algebras include Hecke algebras of finite type, Ariki-Koike algebras, Brauer algebras, Birman-Wenzl algebras and so on. We refer the reader to [2, 4, 12] for details. Recently, Koenig and Xi [7] introduced affine cellular algebras which contain cellular algebras as special cases. They proved affine Hecke algebras of type A are affine cellular.

Several important classes of cellular algebras are symmetric, such as Hecke algebras of finite types, Ariki-Koike algebras and so on. In [8, 9], Li studied the general theory of symmetric cellular algebras, such as dual bases, centres and radicals. Moreover, Li and Xiao considered the classification of the projective cell modules of symmetric cellular agebras in [11]. Frobenius algebras are natural generalizations of symmetric algebras. It is well known that a Frobenius algebra is symmetric if and only if there exists a Nakayama automorphism being the identical mapping. Examples of non-symmetric Frobenius cellular algebras could be found in [6]. In [10], Li investigated Nakayama automorphisms of Frobenius cellular algebras. It is proved that the matrix associated with a Nakayama automorphism with respect to a cellular basis is uni-triangular under a certain condition. A natural question is which properties of symmetric cellular algebras can be generalized to Frobenius cases. In this paper, for a Frobenius cellular algebra, we will give a sufficient and necessary condition for a simple cell module to be projective. A special case that dual bases of the cellular basis satisfying a certain condition is also considered. The result is similar to that in symmetric case.

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The paper is organized as follows. We begin with a quick review on the theory of cellular algebras and Frobenius algebras. In particular, we give in this section a corollary of Gaschütz-Ikeda's Theorem, which is more useful to study cell modules of Frobenius cellular algebras. Then in Section 3, after giving some properties of Frobenius cellular algebras, we give a sufficient and necessary condition for a simple cell module to be projective. A special case will be considered in Section 4. The result is similar to that in symmetric case.

2. Preliminaries

In this section, we will establish the basic notations and some known results which are needed in the following sections. The main references for this section are [3] and [4].

2.1. **Frobenius algebras.** Let R be a commutative ring with identity and A a finite dimensional associative R-algebra. Suppose that there exists an R-bilinear form $f : A \times A \to R$. We say f is *non-degenerate* if the determinant of the matrix $(f(a_i, a_j))_{a_i, a_j \in B}$ is not zero for some basis B of A. We call f *associative* if $f(ab, c) = f(a, bc)$ for all $a, b, c \in A$.

Definition 2.1. An R-algebra A is called *Frobenius* if there is a non-degenerate associative bilinear form f on A.

Let A be a Frobenius algebra with a basis $B = \{a_i \mid i = 1, \dots, n\}$. Let us take a non-degenerate associative bilinear form f. Define an R-linear map $\tau : A \to R$ by

$$\tau(a) = f(a, 1).$$

We call $d = \{d_i \mid i = 1, \dots, n\}$ the *right dual basis* of B which is uniquely determined by the requirement that $\tau(a_i d_j) = \delta_{ij}$ for all $i, j = 1, \dots, n$. Similarly, the *left dual basis* $D = \{D_i \mid i = i, \dots, n\}$ is determined by the requirement that $\tau(D_j a_i) = \delta_{ij}$. Define an R-linear map $\alpha : A \to A$ by

$$\alpha(d_i) = D_i.$$

Then α is a Nakayama automorphism of A.

From now on, all of modules considered in this paper will always be left modules. Let M be an A-module and $\theta \in \mathrm{End}_R(M)$. Then we define the *averaging operator* $I(\theta) \in \mathrm{End}_R(M)$ by

$$I(\theta)(m) := \sum_i a_i \theta(D_i m) \qquad \forall\, m \in M.$$

It is an A-module endomorphism. Furthermore, $I(\theta)$ is independence of the choice of basis. This implies that $I(\theta)$ also can be defined as follows.

$$I(\theta)(m) = \sum_i d_i \theta(a_i m) \qquad \forall\, m \in M.$$

Let $\theta, \pi, \varphi \in \mathrm{End}_A(M)$. Then the definition implies that

$$I(\varphi \circ \theta) = I(\varphi) \circ \theta \quad \text{and} \quad I(\theta \circ \pi) = \theta \circ I(\pi).$$

Moreover, it is helpful to point out that $I(\theta) \in \mathrm{End}_A(M)$.

One of the importance of the averaging operator is that it provides a criterion for an A-module being projective.

Lemma 2.2. (Gaschütz-Ikeda) Let A be a Frobenius algebra and M an A-module. Then M is projective if and only if there exists some $\varphi \in \mathrm{End}_R(M)$ such that $I(\varphi) = \mathrm{id}_M$.

Let R be a field. Suppose that $\dim M = m$ and take a basis $\{v_1, \cdots, v_m\}$. For $1 \leq i, j \leq m$, define $\varphi_{ij} \in \mathrm{End}_R(M)$ by $\varphi_{ij}(v_s) = \delta_{is}v_j$. Clearly, φ_{ij} form a basis of $\mathrm{End}_R(M)$. The following result is a simple corollary of Lemma 2.2. However, it is important in this paper. Hence we will give a complete proof of it here.

Corollary 2.3. *Let A be a finite dimensional Frobenius algebra and M a simple A-module. Then M is projective if and only if there exists some φ_{ij} such that $I(\varphi_{ij}) \neq 0$.*

Proof. Since φ_{ij} form a basis of $\mathrm{End}_R(M)$, the necessity is a direct corollary of Lemma 2.2. Conversely, assume that there exists some φ_{ij} such that $I(\varphi_{ij}) \neq 0$. Note that M is simple. Then Schur's lemma implies that $I(\varphi_{ij}) = r_{ij}\,\mathrm{id}_\mathrm{M}$, where $r_{ij} \in R - \{0\}$. Let $\pi : N \to M$ be an epimorphism of A-modules. Then there exists an R-linear map $\mu : M \to N$ such that $\pi \circ \mu = \mathrm{id}_\mathrm{M}$. This gives that $\pi \circ \mu \circ \varphi_{ij} = \varphi_{ij}$ and thus $I(\pi \circ \mu \circ \varphi_{ij}) = r_{ij}\,\mathrm{id}_\mathrm{M}$. This implies that $r_{ij}^{-1} I(\mu \circ \varphi_{ij})$ is the desired map. □

2.2. **Cellular algebras.** Let us recall the definition of a cellular algebra given by Graham and Lehrer in [4].

Definition 2.4. [4] Let R be a commutative ring with identity. An associative unital R-algebra is called a *cellular algebra* with cell datum (Λ, M, C, i) if the following conditions are satisfied:

(C1) The finite set Λ is a poset. Associated with each $\lambda \in \Lambda$, there is a finite set $M(\lambda)$. The algebra A has an R-basis $\{C^\lambda_{S,T} \mid \lambda \in \Lambda,\ S, T \in M(\lambda)\}$.

(C2) The map i is an R-linear anti-automorphism of A with $i^2 = \mathrm{id}$ which sends $C^\lambda_{S,T}$ to $C^\lambda_{T,S}$ for all $\lambda \in \Lambda$ and $S, T \in M(\lambda)$.

(C3) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, then for any element $a \in A$, we have

$$aC^\lambda_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S) C^\lambda_{S',T} \pmod{\mathrm{A}(< \lambda)},$$

where $r_a(S^{'}, S) \in R$ is independent of T and where $A(< \lambda)$ is the R-submodule of A generated by $\{C^\mu_{U,V} \mid U, V \in M(\mu),\ \mu < \lambda\}$.

If we apply i to the equation in (C3), we obtain
(C3′) $C^\lambda_{T,S} i(a) \equiv \sum\limits_{S' \in M(\lambda)} r_a(S^{'}, S) C^\lambda_{T,S'} \quad (\mod A(< \lambda))$.

As a natural consequence of the axioms, the *cell modules* are defined as follows.

Definition 2.5. Let A be a cellular algebra with cell datum (Λ, M, C, i). For each $\lambda \in \Lambda$, the *cell modules* $W_C(\lambda)$ is left A-module defined as follows: $W_C(\lambda)$ is a free R-module with basis $\{C_S \mid S \in M(\lambda)\}$ and A-action defined by

$$aC_S = \sum_{S' \in M(\lambda)} r_a(S^{'}, S) C_{S'} \quad (a \in A, S \in M(\lambda)),$$

where $r_a(S^{'}, S)$ is the element of R defined in Definition 2.4(C3).

Let A be a cellular algebra with cell datum (Λ, M, C, i). Let $\lambda \in \Lambda$ and $S, T, U, V \in M(\lambda)$. Then we have from Definition 2.4 that

$$C^{\lambda}_{S,T} C^{\lambda}_{U,V} \equiv \Phi(T,U) C^{\lambda}_{S,V} \pmod{\mathrm{A}(< \lambda)},$$

where $\Phi(T,U) \in R$ depends only on T and U. Thus for a cell module $W_C(\lambda)$, we can define a symmetric bilinear form $\Phi_\lambda : W_C(\lambda) \times W_C(\lambda) \longrightarrow R$ by

$$\Phi_\lambda(C_S, C_T) = \Phi(S,T).$$

The radical of the bilinear form is defined to be

$$\operatorname{rad}_\Phi \lambda := \{x \in W(\lambda) \mid \Phi_\lambda(x,y) = 0 \text{ for all } y \in W_C(\lambda)\}.$$

By the general theory of cellular algebras, $\operatorname{rad}_\Phi \lambda$ is an A-submodule of $W_C(\lambda)$, motivating the definition $L(\lambda) = W_C(\lambda)/\operatorname{rad}_\Phi \lambda$. In principle, the next result of Graham and Lehrer classifies the simple A-modules.

Lemma 2.6. ([4]) *Let K be a field and A a finite dimensional cellular algebra over K. Let $\Lambda_0 = \{\lambda \in \Lambda \mid \Phi_\lambda \neq 0\}$. Then $\{L(\lambda) \mid \lambda \in \Lambda_0\}$ is a complete set of pairwise non-isomorphic absolutely irreducible A-modules.*

For any $\lambda \in \Lambda$, fix an order on $M(\lambda)$ and let $M(\lambda) = \{S_1, S_2, \cdots, S_{n_\lambda}\}$, where n_λ is the number of elements in $M(\lambda)$, the matrix

$$G(\lambda) = (\Phi(S_i, S_j))_{1 \leq i,j \leq n_\lambda}$$

is called *Gram matrix*. Note that all the determinants of $G(\lambda)$ defined with different order on $M(\lambda)$ are the same. By the definition of $G(\lambda)$ and $\operatorname{rad}_\Phi \lambda$, for a finite dimensional cellular algebra A, if $\Phi_\lambda \neq 0$, then $\dim_K L(\lambda) = \operatorname{rank} G(\lambda)$.

3. Projective cell modules of Frobenius cellular algebras

Let K be a field. Let A be a finite dimensional K-algebra and M an A-module. The algebra homomorphism

$$\rho_M : A \to \operatorname{End}_K(M), \quad \rho_M(a)m = am, \quad \forall\, m \in M, \quad a \in A,$$

is called the representation afforded by M.

Let A be a finite dimensional Frobenius cellular K-algebra with a cell datum (Λ, M, C, i). Given a non-degenerate bilinear form f, denote the left dual basis by $D = \{D^{\lambda}_{S,T} \mid S, T \in M(\lambda), \lambda \in \Lambda\}$, which satisfies

$$\tau(D^{\mu}_{U,V} C^{\lambda}_{S,T}) = \delta_{\lambda\mu}\delta_{SV}\delta_{TU}.$$

Denote the right dual basis by $d = \{d^{\lambda}_{S,T} \mid S, T \in M(\lambda), \lambda \in \Lambda\}$, which satisfies

$$\tau(C^{\lambda}_{S,T} d^{\mu}_{U,V}) = \delta_{\lambda\mu}\delta_{S,V}\delta_{T,U}.$$

For $\mu \in \Lambda$, let $A_d(> \mu)$ be the K-subspace of A generated by

$$\{d^{\epsilon}_{X,Y} \mid X, Y \in M(\epsilon), \mu < \epsilon\}$$

and let $A_D(> \mu)$ be the K-subspace of A generated by

$$\{D^{\epsilon}_{X,Y} \mid X, Y \in M(\epsilon), \mu < \epsilon\}.$$

Note that the K-linear map α which sends $d^{\epsilon}_{X,Y}$ to $D^{\epsilon}_{X,Y}$ is a Nakayama automorphism of the algebtra A.

For arbitrary λ, $\mu \in \Lambda$, S, $T \in M(\lambda)$, U, $V \in M(\mu)$, write

$$C^\lambda_{S,T}C^\mu_{U,V} = \sum_{\epsilon\in\Lambda, X,Y\in M(\epsilon)} r_{(S,T,\lambda),(U,V,\mu),(X,Y,\epsilon)}C^\epsilon_{X,Y},$$

$$D^\lambda_{S,T}D^\mu_{U,V} = \sum_{\epsilon\in\Lambda, X,Y\in M(\epsilon)} R_{(S,T,\lambda),(U,V,\mu),(X,Y,\epsilon)}D^\epsilon_{X,Y}.$$

Applying α^{-1} on both sides of the above equation, we obtain

$$d^\lambda_{S,T}d^\mu_{U,V} = \sum_{\epsilon\in\Lambda, X,Y\in M(\epsilon)} R_{(S,T,\lambda),(U,V,\mu),(X,Y,\epsilon)}d^\epsilon_{X,Y}.$$

The following lemma about structure constants will play an important role in determining the projective cell modules of Frobenius cellular algebras.

Lemma 3.1. *For arbitrary $\lambda, \mu \in \Lambda$ and $S, T, P, Q \in M(\lambda)$, $U, V \in M(\mu)$ and $a \in A$, the following hold:*

(1) $D^\mu_{U,V}C^\lambda_{S,T} = \sum_{\epsilon\in\Lambda, X,Y\in M(\epsilon)} r_{(S,T,\lambda),(Y,X,\epsilon),(V,U,\mu)}D^\epsilon_{X,Y}$.

(2) $D^\mu_{U,V}C^\lambda_{S,T} = \sum_{\epsilon\in\Lambda, X,Y\in M(\epsilon)} R_{(Y,X,\epsilon),(U,V,\mu),(T,S,\lambda)}C^\epsilon_{X,Y}$.

(3) $aD^\mu_{U,V} \equiv \sum_{U'\in M(\mu)} r_{i(\alpha^{-1}(a))}(U,U')D^\mu_{U',V} \pmod{A_D(>\mu)}$.

(4) $D^\lambda_{P,Q}C^\lambda_{S,T} = 0$ if $Q \neq S$.

(5) $D^\mu_{U,V}C^\lambda_{S,T} = 0$ if $\mu \nleq \lambda$.

(6) $D^\lambda_{T,S}C^\lambda_{S,Q} = D^\lambda_{T,P}C^\lambda_{P,Q}$.

(7) $C^\lambda_{S,T}d^\mu_{U,V} = \sum_{\epsilon\in\Lambda, X,Y\in M(\epsilon)} r_{(Y,X,\epsilon),(S,T,\lambda),(V,U,\mu)}d^\epsilon_{X,Y}$.

(8) $C^\lambda_{S,T}d^\mu_{U,V} = \sum_{\epsilon\in\Lambda, X,Y\in M(\epsilon)} R_{(U,V,\mu),(Y,X,\epsilon),(T,S,\lambda)}C^\epsilon_{X,Y}$.

(9) $d^\mu_{U,V}a \equiv \sum_{V'\in M(\mu)} r_{\alpha(a)}(V,V')d^\mu_{U,V'} \pmod{A_d(>\mu)}$.

(10) $C^\lambda_{S,T}d^\lambda_{P,Q} = 0$ if $T \neq P$.

(11) $C^\lambda_{S,T}d^\mu_{U,V} = 0$ if $\mu \nleq \lambda$.

(12) $C^\lambda_{S,T}d^\lambda_{T,P} = C^\lambda_{S,Q}d^\lambda_{Q,P}$.

Proof. (1), (3), (4), (5), (7), (8), (9), (11) have been obtained in [10]. (2), (8) are proved similarly as (1), (7), respectively. (6) is a direct corollary of (1) and Definition 2.4. Finally, (12) is obtained similarly as (6). □

It follows from Definition 2.4 and Lemma 3.1 (1), (3) that for arbitrary elements $S, T, P, Q \in M(\lambda)$,

$$D^\lambda_{S,T}D^\lambda_{P,Q} \equiv \Psi(T,P)D^\lambda_{S,Q} \pmod{\mathrm{A_D}(>\lambda)},$$

where $\Psi(T,P) \in K$ depends only on T and P. Applying α^{-1} on both sides of the above equation, we obtain

$$d^\lambda_{S,T}d^\lambda_{P,Q} \equiv \Psi(T,P)d^\lambda_{S,Q} \pmod{\mathrm{A_d}(>\lambda)}.$$

Take an order on $M(\lambda)$ the same as in the definition of $G(\lambda)$. Then the matrix $(\Psi(S,T))$ will be denoted by $G'(\lambda)$.

Let $W_C(\lambda)$ be a cell module. Define $\varphi_{ST} \in \mathrm{End}_K(W_C(\lambda))$ by $\varphi_{ST}(C_X) = \delta_{SX}C_T$ for arbitrary $S, T \in M(\lambda)$. The following lemma reveals a relation among $I(\varphi_{ST})$, Φ_λ and Ψ_λ.

Lemma 3.2. *If $W_C(\lambda)$ is simple, then $I(\varphi_{ST}) = c_{ST}\mathrm{id}_{W_C(\lambda)}$, where*

$$c_{ST} = \sum_{Q\in M(\lambda)} \Phi(T,Q)\Psi(S,Q).$$

Proof. It follows from $W_C(\lambda)$ being simple and Schur's Lemma that

$$I(\varphi_{ST}) = c_{ST}\mathrm{id}_{W_C(\lambda)}, \tag{3.1}$$

where $c_{ST} \in K$. Let ρ_λ be the representation afforded by $W_C(\lambda)$. Then for $a \in A$, we have

$$aC_X = \sum_{Y\in M(\lambda)} \rho_\lambda(a)_{YX}C_Y.$$

A direct computation gives that

$$I(\varphi_{ST})(C_X) = \sum_{\eta\in\Lambda,\ P,\ Q\in M(\eta),\ Y\in M(\lambda)} \rho_\lambda(D^\eta_{Q,P})_{SX}\rho_\lambda(C^\eta_{P,Q})_{YT}C_Y. \tag{3.2}$$

Combining (3.1) and (3.2) yields that

$$\sum_{\eta\in\Lambda,P,Q\in M(\eta),Y\in M(\lambda)} \rho_\lambda(D^\eta_{Q,P})_{SX}\rho_\lambda(C^\eta_{P,Q})_{XT} = c_{ST}. \tag{3.3}$$

By the definition of cell modules,

$$C^\eta_{P,Q}C_T = r_{C^\eta_{P,Q}}(X,T)C_X + \sum_{X'\neq X} r_{X'}C_{X'},$$

where $r_{C^\eta_{P,Q}}(X,T) \in K$ is defined in Definition 2.1(C3) and $r_{X'} \in K$. This implies that

$$\rho_\lambda(C^\eta_{P,Q})_{XT} = r_{(P,Q,\eta),(T,T,\lambda),(X,T,\lambda)}. \tag{3.4}$$

On the other hand, it follows from Lemma 3.1(2) that

$$\rho_\lambda(D^\eta_{Q,P})_{SX} = R_{(S,S,\lambda),(Q,P,\eta),(S,X,\lambda)}. \tag{3.5}$$

We have from (3.3), (3.4) and (3.5) that

$$c_{ST} = \sum_{\eta\in\Lambda,P,Q\in M(\eta)} r_{(P,Q,\eta),(T,T,\lambda),(X,T,\lambda)}R_{(S,S,\lambda),(Q,P,\eta),(S,X,\lambda)}. \tag{3.6}$$

Again by Lemma 3.1, for any $\epsilon \in \Lambda$, if $\epsilon \nleq \lambda$, then $R_{(S,S,\lambda),(Y,X,\epsilon),(S,S,\lambda)} = 0$, if $\lambda \nleq \epsilon$, then $r_{(X,Y,\epsilon),(S,S,\lambda),(S,S,\lambda)} = 0$. Thus Definition 2.4 and Lemma 3.1 force (3.6) being

$$\begin{aligned} c_{ST} &= \sum_{P,Q\in M(\lambda)} r_{(P,Q,\lambda),(T,T,\lambda),(X,T,\lambda)}R_{(S,S,\lambda),(Q,P,\lambda),(S,X,\lambda)} \\ &= \sum_{Q\in M(\lambda)} r_{(X,Q,\lambda),(T,T,\lambda),(X,T,\lambda)}R_{(S,S,\lambda),(Q,X,\lambda),(S,S,\lambda)} \\ &= \sum_{Q\in M(\lambda)} \Phi(T,Q)\Psi(S,Q). \end{aligned}$$

We complete the proof. □

Now we are able to describe the main result of this section.

Theorem 3.3. *Let A be a finite dimensional Frobenius cellular K-algebra and $W_C(\lambda)$ a simple cell module. Then $W_C(\lambda)$ is projective if and only if there exist $S, T \in M(\lambda)$ such that $\Psi(S,T) \neq 0$.*

Proof. It follows from Corollary 2.3 that $W_C(\lambda)$ is projective if and only if there exist $X, Y \in M(\lambda)$ such that $I(\varphi_{XY}) \neq 0$. By Lemma 3.2, this is equivalent to $c_{XY} \neq 0$. Fix an order on $M(\lambda)$ and denote the matrix (c_{XY}) by I_λ. Then $W_C(\lambda)$ is projective if and only if $I_\lambda \neq 0$. On the other hand, we have from Lemma 3.2 that $G'(\lambda)G(\lambda) = I_\lambda$. The simplicity of $W_C(\lambda)$ implies that $G(\lambda)$ is invertible. Thus $I_\lambda \neq 0$ if and only if $G'(\lambda) \neq 0$, that is, there exist $S, T \in M(\lambda)$ such that $\Psi(S,T) \neq 0$. □

We can obtain from this theorem a necessary condition for a cell module being projective.

Corollary 3.4. *If $G'(\lambda) = 0$, then $W_C(\lambda)$ is not a projective module.*

Proof. The corollary follows from Theorem 3.3 and [7, Corollary 1.2] clearly. □

Remark 3.5. Using the right dual basis $\{d^\epsilon_{X,Y} \mid \epsilon \in \Lambda, X, Y \in M(\epsilon)\}$, for each $\lambda \in \Lambda$ we can define an A-module $W_d(\lambda)$ as follows. As a K-basis, $W_d(\lambda)$ has a basis $\{d_S \mid S \in M(\lambda)\}$. The A-action is defined by

$$ad_S = \sum_{S' \in M(\lambda)} r_{i(a)}(S, S')d_{s'}.$$

Then by a similar way, we can prove that if $W_d(\lambda)$ is simple, then it is projective if and only if $\lambda \in \Lambda_0$.

Now let us apply Theorem 3.3 to an example which was constructed by König and Xi in [6].

Example 3.6. Let K be a field. Let us take $\lambda \in K$ with $\lambda \neq 0$ and $\lambda \neq 1$. Let

$$A = K\langle a, b, c, d\rangle / I,$$

where I is generated by

$$a^2, b^2, c^2, d^2, ab, ac, ba, bd, ca, cd, db, dc, cb - \lambda bc, ad - bc, da - bc.$$

Let $\Lambda = \{1, 2, 3\}$. If we define τ by $\tau(1) = \tau(a) = \tau(b) = \tau(c) = \tau(d) = 0$ and $\tau(bc) = 1$ and define an involution i on A to be fixing a and d, but interchanging b and c, then A is a Frobenius cellular algebra with a cellular basis

$$bc; \quad \begin{matrix} a & b \\ c & d \end{matrix}; \quad 1.$$

The left dual basis is

$$1; \quad \begin{matrix} d & c/\lambda \\ b & a \end{matrix}; \quad bc.$$

It is easy to check that $W_C(1) = 0$, $W_C(2) = 0$ and $W_C(3)$ is simple. However, $G'(3) = 0$. This implies that $W_C(3)$ is not projective. Thus none of cell modules is projective.

4. A special case

Throughout this section, we assume that A is a finite dimensional Frobenius cellular algebra with both the dual bases being cellular with respect to the opposite order on Λ.

Under this assumption, we obtained the following result about the Nakayama automorphism α in [10].

Lemma 4.1. [10, Theorem 3.1] *For arbitrary $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, we have*

$$\alpha(C^{\lambda}_{S,T}) \equiv C^{\lambda}_{S,T} \pmod{A(<\lambda)}.$$

In order to generalize the so-called Schur elements to Frobenius case, we prove a lemma first.

Lemma 4.2. *Let A be a Frobenius cellular algebra with cell datum (Λ, M, C, i). For every $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, we have*

$$D^{\lambda}_{S,T}C^{\lambda}_{T,S}D^{\lambda}_{S,T}C^{\lambda}_{T,S} = \sum_{S' \in M(\lambda)} \Phi(S', T)\Psi(S', T)D^{\lambda}_{S,T}C^{\lambda}_{T,S}.$$

Proof. By Lemma 3.1 (3), (5), we have

$$\begin{aligned} D^{\lambda}_{S,T}C^{\lambda}_{T,S}D^{\lambda}_{S,T}C^{\lambda}_{T,S} &= D^{\lambda}_{S,T}(C^{\lambda}_{T,S}D^{\lambda}_{S,T})C^{\lambda}_{T,S} \\ &= \sum_{S' \in M(\lambda)} r_{i(\alpha^{-1}(C^{\lambda}_{T,S}))}(S, S')D^{\lambda}_{S,T}D^{\lambda}_{S',T}C^{\lambda}_{T,S}. \end{aligned}$$

It follows from Definition 2.4 and Lemma 4.1 that

$$r_{i(\alpha^{-1}(C^{\lambda}_{T,S}))}(S, S') = \Phi(S', T).$$

Again by Lemma 3.1(5), the desired equation follows. □

We have from Lemma 3.1 that $D^{\lambda}_{S,T}C^{\lambda}_{T,S}$ is independent of T. Then for any $\lambda \in \Lambda$, we can define a constant k_λ as follows.

Definition 4.3. For $\lambda \in \Lambda$, take an arbitrary $T \in M(\lambda)$. Define

$$k_\lambda = \sum_{X \in M(\lambda)} \Phi(X, T)\Psi(X, T).$$

The following lemma reveals the relation among $G(\lambda)$, $G'(\lambda)$ and k_λ. This result is a bridge which connects k_λ with c_{ST}.

Lemma 4.4. *Let $\lambda \in \Lambda$. Fix an order on the set $M(\lambda)$. Then $G(\lambda)G'(\lambda) = k_\lambda E$, where E is the identity matrix.*

Proof. Clearly, we only need to show that $\sum_{X \in M(\lambda)} \Phi(X, S)\Psi(X, T) = 0$ for arbitrary $S, T \in M(\lambda)$ with $S \neq T$.

Let us consider $D^{\lambda}_{S,S}C^{\lambda}_{S,S}D^{\lambda}_{S,S}C^{\lambda}_{T,S}$. On one hand, it is zero by Lemma 3.1. On the other hand, computing it as that in Lemma 4.2 yields that

$$D^{\lambda}_{S,S}C^{\lambda}_{S,S}D^{\lambda}_{S,S}C^{\lambda}_{T,S} = \sum_{X \in M(\lambda)} \Phi(X, S)\Psi(X, T)D^{\lambda}_{S,S}C^{\lambda}_{S,S}.$$

It follows from $D^{\lambda}_{S,S}C^{\lambda}_{S,S} \neq 0$ that $\sum_{X \in M(\lambda)} \Phi(X, S)\Psi(X, T) = 0$. □

Corollary 4.5. *Keep notations as above, then* $c_{ST} = \mathrm{Tr}(\varphi_{ST})k_\lambda$, *where* Tr *denotes the usual matrix trace.*

Proof. Note that $\mathrm{Tr}(\varphi_{ST}) = \delta_{ST}$. Then by Lemma 3.2, Definition 4.3 and Lemma 4.4, the corollary follows. □

The constants k_λ could be viewed as generalizations of Schur elements when $W_C(\lambda)$ is simple. We are in a position to give the main result of this section. It is similar to that in symmetric case.

Theorem 4.6. *Let* A *be a Frobenius cellular algebra with* $i(D^\lambda_{S,T}) = D^\lambda_{T,S}$ *and* $i(d^\lambda_{S,T}) = d^\lambda_{T,S}$ *for arbitrary* $\lambda \in \Lambda$ *and* $S, T \in M(\lambda)$. *Then the cell module* $W_C(\lambda)$ *is projective if and only if* $k_\lambda \neq 0$.

Proof. By [6, Corollary 1.2], if $W_C(\lambda)$ is projective, then $W_C(\lambda)$ is irreducible. It follows from Corollary 2.3 that there exist $S, T \in M(\lambda)$ such that $I(\varphi_{ST}) \neq 0$, or $c_{ST} \neq 0$. This implies that $k_\lambda \neq 0$ by Corollary 4.5.

Conversely, assume that $k_\lambda \neq 0$. Then $W_C(\lambda)$ is irreducible by Lemma 4.4. Again by Corollary 2.3, $W_C(\lambda)$ is projective. □

Corollary 4.7. $W_C(\lambda)$ *is projective if and only if so is* $W_d(\lambda)$.

Proof. By Theorem 4.6, we only need to prove that $W_d(\lambda)$ is projective if and only if $k_\lambda \neq 0$. In fact, if $k_\lambda \neq 0$, then Lemma 4.4 forces $W_d(\lambda)$ to be simple. Thus $W_d(\lambda)$ being projective follows. Conversely, if $W_d(\lambda)$ is projective, then [6, Corollary 1.2] implies that $W_d(\lambda)$ is simple, that is, $G'(\lambda)$ is invertible. Moreover, we have from Remark 3.5 that $G(\lambda) \neq 0$. Hence we conclude by Lemma 4.4 that $k_\lambda \neq 0$. □

Remark 4.8. Using the left dual basis $D^\epsilon_{X,Y}$, we can also define modules $W_D(\lambda)$. It could be proved that $W_D(\lambda)$ is projective if and only if so is $W_d(\lambda)$. We omit the details here.

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